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INTERACTION OF SLOWLY MOVING PUNCHES WITH AN ELASTIC HALF-SPACE

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The problem of slow dynamic contact interaction of a system of punches remote from each other with an elastic half-space surface in the absence of friction is studied under the assumptions that the diameters of the contact areas are smaller than the minimum distance between the punches and the time required for the shear wave to travel the distance equal to the punch diameter is comparable to the time scale of the process. A first-order asymptotic model is constructed. As an example, the case of steady-state vibrations of a system of two punches is considered.

Key words: *vibrations, punches, elastic half-space, Lamb problem, asymptotic methods.*

1. Formulation of the Problem. Let points $P^1,\ldots,P^N$ with the coordinates (x_1^j,x_2^j), $j=1,2,\ldots,N$, be distinguished on the boundary of an elastic half-space (with Young's modulus E and Poisson's ratio ν). The smallest possible distance $d_{jk}=|P^j-P^k|$ $(j\neq k)$ will be denoted by d.

Let us consider the dynamic contact problem for a system of $N\geq 2$ punches which have centers at the given points and occupy in plan the regions $\omega_\varepsilon^1,\ldots,\omega_\varepsilon^N$ of small (of order εd) diameter. Here and below, $\varepsilon>0$ is a small dimensionless parameter. The region ω_ε^j is obtained by compressing a certain fixed plane region ω_1^j by a factor of ε^{-1}:

$$\omega_\varepsilon^j=\{(x_1,x_2):\quad \varepsilon^{-1}(x_1-x_1^j,x_2-x_2^j)\in\omega_1^j\}.$$

For simplicity, we will assume that the region ω_1^j in the plane of the stretched coordinates

$$(\xi_1^j,\xi_2^j)=\varepsilon^{-1}(x_1-x_1^j,x_2-x_2^j)$$

is contained in a circle of radius $d/2$ with center at the coordinate origin. Then, for any values of the parameter $\varepsilon\in(0,1)$, the regions $\omega_\varepsilon^1,\ldots,\omega_\varepsilon^N$ do not have self-intersections.

We assume that the punches have flat bottoms and the close contact of the bottoms to the elastic foundation surface is retained throughout the motion. We denote the specified vertical displacement of the center of the punch as $\delta_0^j(t)$ and the angles of rotation of the punch around the axes through its center and parallel to the coordinate axes Ox_1 and Ox_2 as $\beta_1^j(t)$ and $\beta_2^j(t)$, respectively.

It is assumed that the friction under the bottoms of the punches is negligibly small. Then, the contact pressure densities $p^1(t,x_1,x_2),\ldots,p^N(t,x_1,x_2)$ on the boundary of the elastic half-space, according to the solution of Lamb problem, satisfy the following system of integral equations [1, 2]:

$$\sum_{k=1}^N (D_\varepsilon^k p^k)(t,x_1,x_2)=\delta_0^j-\beta_2^j(x_1-x_1^j)+\beta_1^j(x_2-x_2^j),\qquad (x_1,x_2)\in\omega_\varepsilon^j. \tag{1.1}$$

Here $j=1,2,\ldots,N$ and D_ε^k is a linear integral operator

$$(D_\varepsilon^k p^k)(x_1,x_2)=\int_0^t\iint_{\omega_\varepsilon^k}\frac{\partial}{\partial(t-\tau)}\,G_3(t-\tau,|\boldsymbol{x}-\boldsymbol{y}|)p^k(\tau,\boldsymbol{y})\,d\boldsymbol{y}\,d\tau, \tag{1.2}$$

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where $G_3(t, r)$ is the vertical displacement of points of the surface of the elastic half-space caused by the action of a constant unit concentrated force suddenly applied to the boundary at the coordinate origin (with polar radius $r = 0$) at the time $t = 0$.

The following representation is valid [3]:

$$G_3(t,r) = T_3(r)[H(t - r/c_1)/2 + H(t - r/c_2)/2$$

$$+ f_{12}(c_2t/r)H(t - r/c_1)H(r/c_2 - t) - f_{23}(c_2t/r)H(t - r/c_2)H(r/c_3 - t)]. \quad (1.3)$$

Here $H(t)$ is a Heaviside function, c_1 and c_2 are the tensile- and shear-wave velocities, c_3 is the Rayleigh-wave velocity, $T_3(r) = (1 - \nu^2)(\pi E)^{-1}r^{-1}$ is the vertical displacement of points of the surface of the elastic half-space in the Boussinesq problem, and the functions $f_{12}(c_2t/r)$ and $f_{23}(c_2t/r)$ are defined by the formulas

$$f_{12}(c_2t/r) = -q_1(c_2^2t^2/r^2 - p_1)^{-1/2} + q_2(c_2^2t^2/r^2 - p_2)^{-1/2} - q_3(p_3 - c_2^2t^2/r^2)^{-1/2},$$

$$f_{23}(c_2t/r) = 2q_3(p_3 - c_2^2t^2/r^2)^{-1/2},$$

where q_1, q_2, and q_3 are the coefficients given by

$$q_1 = \frac{p_1(\alpha - p_1)\sqrt{1 - p_1}}{2(p_1 - p_2)(p_1 - p_3)}, \qquad q_2 = \frac{p_2(\alpha - p_2)\sqrt{1 - p_2}}{2(p_1 - p_2)(p_2 - p_3)},$$

$$q_3 = \frac{p_3(\alpha - p_3)\sqrt{p_3 - 1}}{2(p_1 - p_3)(p_2 - p_3)}, \qquad \alpha = \frac{c_2^2}{c_1^2} = \frac{1 - 2\nu}{2(1 - \nu)},$$

and p_1, p_2, and p_3 are roots of the equation

$$p^3 - (2 - \nu)p^2 + (1 - \nu)p - (1 - \nu)/8 = 0. \quad (1.4)$$

For $0 < \nu < \nu_*$, p_3 is taken to be the largest of the roots, and for $\nu_* < \nu < 1/2$, it is the single real root $p_3 = c_2^2/c_3^2$. In this case,

$$\nu_* = (1/24)(388 + 114\sqrt{114})^{1/3} - (13/3)(388 + 114\sqrt{114})^{-1/3} + 1/6 \approx 0.263.$$

In the case $\nu = \nu_*$, Eq. (1.4) has multiple roots. In the present paper, this case is not considered.

For $\nu = 1/4$ (this value is commonly used in seismology [4]), we have $q_1 = \sqrt{3}/12$, $q_2 = \sqrt{3\sqrt{3} - 5}/12$, $q_3 = \sqrt{3\sqrt{3} + 5}/12$, $p_1 = 1/4$, $p_2 = (3 - \sqrt{3})/4$, $p_3 = 3 + \sqrt{3}/4$, and $\alpha = 1/3$.

We set

$$G_3(t, r) = T_3(r) + g_3(t, r). \quad (1.5)$$

Substitution of expression (1.5) into (1.2) yields [5]

$$G_3(t, |\boldsymbol{x} - \boldsymbol{y}|) * \frac{\partial p^k}{\partial t}(t, \boldsymbol{y}) = T_3(|\boldsymbol{x} - \boldsymbol{y}|)[p^k(t, \boldsymbol{y}) - p^k(0, \boldsymbol{y})] + g_3(t, |\boldsymbol{x} - \boldsymbol{y}|) * \frac{\partial p^k}{\partial t}(t, \boldsymbol{y}),$$

where

$$G_3(t, |\boldsymbol{x} - \boldsymbol{y}|) * \frac{\partial p^k}{\partial t}(t, \boldsymbol{y}) = \int_0^t G_3(t - \tau, |\boldsymbol{x} - \boldsymbol{y}|)\, \frac{\partial p^k}{\partial \tau}(\tau, \boldsymbol{y})\, d\tau.$$

Thus, formula (1.2) becomes

$$(D_\varepsilon^k p^k)(t, \boldsymbol{x}) = \iint_{\omega_\varepsilon} g_3(t, |\boldsymbol{x} - \boldsymbol{y}|)p^k(0, \boldsymbol{y})\, d\boldsymbol{y} + \iint_{\omega_\varepsilon^k} T_3(|\boldsymbol{x} - \boldsymbol{y}|)p^k(t, \boldsymbol{y})\, d\boldsymbol{y}$$

$$+ \iint_{\omega_\varepsilon^k} g_3(t, |\boldsymbol{x} - \boldsymbol{y}|) * \frac{\partial p^k}{\partial t}(t, \boldsymbol{y})\, d\boldsymbol{y}. \quad (1.6)$$

Equations (1.1) should be supplemented by initial conditions that define the positions and velocities of the punches at $t = 0$. At the initial time, if the elastic half-space is not deformed $[p^j(0, x_1, x_2) \equiv 0]$, we can set

$\delta_0^j(0) = \beta_1^j(0) = \beta_2^j(0) = 0$ and $\dot{\delta}_0^j(0) = \dot{\beta}_1^j(0) = \dot{\beta}_2^j(0) = 0$ ($j = 1, 2, \ldots, N$). However, the method described below to approximately solve the dynamic contact problem does not allow a description of the initial stage of the movement, which is greatly dependent on the initial conditions.

2. Influence of Punches Remote from Each Other. Following the approach used in [6], we distinguish any punch by denoting it with j and write Eq. (1.1) as

$$(D_\varepsilon^j p^j)(t, \boldsymbol{x}) = \delta_0^j - \beta_2^j(x_1 - x_1^j) + \beta_1^j(x_2 - x_2^j) - \sum_{k \neq j} (D_\varepsilon^k p^k)(t, \boldsymbol{x}). \tag{2.1}$$

Using the relation $|\boldsymbol{x} - \boldsymbol{y}|^{-1} = d_{jk}^{-1} + O(\varepsilon)$, we obtain

$$\frac{\pi E}{1 - \nu^2} \iint\limits_{\omega_\varepsilon^k} T_3(|\boldsymbol{x} - \boldsymbol{y}|) p^k(t, \boldsymbol{y})\, d\boldsymbol{y} = \frac{F_3^{\varepsilon k}(t)}{d_{jk}} + \ldots . \tag{2.2}$$

Here $F_3^{\varepsilon k}$ is the resultant given by the formula

$$F_3^{\varepsilon k} = \iint\limits_{\omega_\varepsilon^k} p^k(t, \boldsymbol{y})\, d\boldsymbol{y}.$$

Expressions for terms of order ε and ε^2 in expansion (2.2) are given in [6].

We consider the third integral on the right side of (1.6). According to representations (1.3) and (1.5), some time after the beginning of the movement, the following relation is valid:

$$\begin{aligned} -T_3(|\boldsymbol{x} - \boldsymbol{y}|)^{-1} g_3(t, |\boldsymbol{x} - \boldsymbol{y}|) * \frac{\partial p^k}{\partial t}(t, \boldsymbol{y}) = -\int\limits_0^{c_1^{-1}|\boldsymbol{x}-\boldsymbol{y}|} \frac{\partial p^k}{\partial (t - \tau)}(t - \tau, \boldsymbol{y})\, d\tau \\ + \int\limits_{c_1^{-1}|\boldsymbol{x}-\boldsymbol{y}|}^{c_2^{-1}|\boldsymbol{x}-\boldsymbol{y}|} \Big\{ -\frac{1}{2} + f_{12}\Big(\frac{c_2 \tau}{|\boldsymbol{x} - \boldsymbol{y}|}\Big) \Big\} \frac{\partial p^k}{\partial (t - \tau)}(t - \tau, \boldsymbol{y})\, d\tau \\ - \int\limits_{c_2^{-1}|\boldsymbol{x}-\boldsymbol{y}|}^{c_3^{-1}|\boldsymbol{x}-\boldsymbol{y}|} f_{23}\Big(\frac{c_2 \tau}{|\boldsymbol{x} - \boldsymbol{y}|}\Big) \frac{\partial p^k}{\partial (t - \tau)}(t - \tau, \boldsymbol{y})\, d\tau. \end{aligned} \tag{2.3}$$

Similarly to (2.2), we obtain the expansion

$$\iint\limits_{\omega_\varepsilon^k} g_3(t, |\boldsymbol{x} - \boldsymbol{y}|) * \frac{\partial p^k}{\partial t}(t, \boldsymbol{y})\, d\boldsymbol{y} = g_3(t, d_{jk}) * \frac{dF_3^{\varepsilon k}}{dt}(t) - \ldots , \tag{2.4}$$

which, as the initial representation (2.3), is valid only for $t > c_3^{-1}(d_{jk} + d)$.

3. Slow Movement of an Isolated Punch. We consider the integral on the left side of Eq. (2.1) for $\boldsymbol{x} \in \omega_\varepsilon^j$ and $\boldsymbol{y} \in \omega_\varepsilon^j$. The following expansion is valid [5]:

$$\frac{\pi E}{1 - \nu^2} c_2 g_3(t, |\boldsymbol{x} - \boldsymbol{y}|) * \frac{\partial p^j}{\partial t}(t, \boldsymbol{y}) = -\Lambda_0 \frac{\partial p^j}{\partial t}(t, \boldsymbol{y}) + \ldots . \tag{3.1}$$

Here

$$\Lambda_0 = \frac{1}{2} + \frac{1}{2}\beta^{-1} - \int\limits_{\beta^{-1}}^{1} f_{12}(\tau)\, d\tau + \int\limits_1^{\gamma} f_{23}(\tau)\, d\tau, \tag{3.2}$$

$\beta = c_1/c_2 = \alpha^{-1/2}$ and $\gamma = c_2/c_3 = p_3^{1/2}$. Thus, for the case $t > c_3^{-1}\varepsilon d$, where representation (2.3) is valid and the first term on the right side of (1.6) vanishes, in view of expansion (3.1), we have

$$\frac{\pi E}{1 - \nu^2} (D_\varepsilon^j p^j)(t, \boldsymbol{x}) = \iint\limits_{\omega_\varepsilon^j} |\boldsymbol{x} - \boldsymbol{y}|^{-1} p^j(t, \boldsymbol{y})\, d\boldsymbol{y} - \frac{\Lambda_0}{c_2} \frac{dF_3^{\varepsilon j}}{dt}(t) + \ldots . \tag{3.3}$$

Let an area ω_ε (here and below, the subscript j is omitted for simplicity) be subjected to a time-dependent pressure uniformly distributed over the area $p(t, \boldsymbol{x}) \equiv p_0(t)$ [the function $p_0(t)$ does not depend on the parameter ε]. Then, transforming to stretched coordinates on the right side of relation (3.3), it is easy to show that, as $\varepsilon \to 0$, the first term is of order $\varepsilon p_0(t)$, the second of order $\varepsilon^2 \dot{p}_0(t)$, and the third of order $\varepsilon^3 \ddot{p}_0(t)$ (the point denotes differentiation with respect to time).

In the contact problem for an isolated punch, we can choose the diameter $d_\varepsilon = \varepsilon d$ of the region ω_ε as the unit length, and the quantity $T_\varepsilon = c_3^{-1} \varepsilon d$ as the unit time. In this case, $\tau = t/T_\varepsilon$ is a dimensionless time variable. In conversion to the dimensionless variable τ, the terms on the right side of relation (3.3) have the following orders: $\varepsilon p_0(\varepsilon, \tau)$, $\varepsilon \partial p_0(\varepsilon, \tau)/\partial \tau$, and $\varepsilon \partial^2 p_0(\varepsilon, \tau)/\partial \tau^2$. Hence, in order that expansion (3.3) be asymptotic, it is necessary that the contact pressure changes only slightly during the time the Rayleigh wave travels a distance equal to the punch diameter. For a particular law of pressure variation given by a fairly smooth function $p_0(t)$, this requirement is satisfied for a punch occupying in plan an area of small diameter ω_ε, i.e., for $\varepsilon \ll 1$.

For simplicity, we confine ourselves to the first correction to the quasistatic approximation (3.3). Denoting the displacement of the punch by $\delta_0(t)$, we obtain the following equation for the slow vertical movement of an isolated punch:

$$(B_\varepsilon p)(t, \boldsymbol{x}) = \delta_0(t) + \frac{\Lambda_0}{c_2} \frac{1 - \nu^2}{\pi E} \dot{F}_3^\varepsilon(t), \qquad \boldsymbol{x} \in \omega_\varepsilon. \tag{3.4}$$

Here B_ε is the linear integral operator of the static contact problem:

$$(B_\varepsilon p)(t, \boldsymbol{x}) = \frac{1 - \nu^2}{\pi E} \iint\limits_{\omega_\varepsilon} \frac{p(t, \boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|} \, d\boldsymbol{y}. \tag{3.5}$$

Since the right side of Eq. (3.4) does not depend on the coordinates x_1 and x_2, the following relation holds:

$$F_3^\varepsilon(t) = \frac{\pi E}{1 - \nu^2} \, c^\varepsilon \Big(\delta_0(t) + \frac{\Lambda_0}{c_2} \frac{1 - \nu^2}{\pi E} \, \dot{F}_3^\varepsilon(t) \Big). \tag{3.6}$$

Here c^ε is the sliding capacity of a punch with a flat smooth bottom (see, for example, [7–9]). For an elliptic punch with major half-axis $a^\varepsilon = \varepsilon a$ and eccentricity e, we have [10]

$$c^\varepsilon = a^\varepsilon / K(e), \tag{3.7}$$

where $K(e)$ is a complete elliptic integral of the first kind.

Equation (3.6) is written as

$$-\frac{\Lambda_0}{c_2} \, \dot{F}_3^\varepsilon(t) + \frac{1}{c^\varepsilon} \, F_3^\varepsilon(t) = \frac{\pi E}{1 - \nu^2} \, \delta_0(t). \tag{3.8}$$

Setting $F_3^\varepsilon(t) = F_{30} \exp(i\omega t)$, from Eq. (3.8) we obtain $\delta_0(t) = \delta_{00} \exp(i\omega t)$, where

$$\delta_{00} = \frac{1 - \nu^2}{\pi E} \Big(\frac{1}{c^\varepsilon} - i\omega \, \frac{\Lambda_0}{c_2} \Big) F_{30}.$$

Thus, the impedance defined as the ratio of the force amplitude F_{30} to the displacement amplitude δ_{00} [11] is equal to

$$\frac{F_{30}}{\delta_{00}} = \frac{1 - \nu^2}{\pi E} \, c^\varepsilon \Big(1 - i\omega c^\varepsilon \, \frac{\Lambda_0}{c_2} \Big)^{-1}. \tag{3.9}$$

In the examined case with the fixed diameter d_ε^j of the punch bottom ω_ε, the assumption of slow movement of the punch, made in the derivation of Eq. (3.4), implies that the relative frequency of the vibrations is low. Thus, with the same accuracy as Eq. (3.6) was derived, from relation (3.9) we find that

$$\frac{F_{30}}{\delta_{00}} = \frac{1 - \nu^2}{\pi E} \, c^\varepsilon \Big(1 + i\lambda_0 \, \frac{\omega d_\varepsilon}{2c_2} + O(\varepsilon^2) \Big), \tag{3.10}$$

where $\lambda_0 = 2c^\varepsilon \Lambda_0 / d_\varepsilon$.

For comparison with known results for a circular punch [11], we calculate the quantity $\lambda_0 = 2\pi^{-1} \Lambda_0$. For $\nu = 1/4$, we have $\lambda_0 = 0.79$, which agrees with calculation results [11].

We note that, in comparison with expansion (3.10), formula (3.9), which can be treated as a simple Pade approximation, is more accurate [12, 13].

4. Asymptotic Model of Slow Vertical Movement of a System of Punches Remote from Each Other. On the left side of Eq. (2.1), we substitute the approximate expression for the operator D_ε^j taking into account the two first terms of formula (3.3), and on the right side, the approximate expression for the operator D_ε^k according to representation (1.6) and taking into account the first terms in formulas (2.2) and (2.4). As a result, we obtain the equation

$$(B_\varepsilon^j p^j)(t,\boldsymbol{x}) = \delta_0^j(t) + \frac{\Lambda_0}{c_2}\,\frac{1-\nu^2}{\pi E}\,\dot F_3^{\varepsilon j}(t) - \sum_{k\neq j}\Big(\frac{1-\nu^2}{\pi E}\,\frac{F_3^{\varepsilon k}(t)}{d_{jk}} + g_3(t,d_{jk}) * \dot F_3^{\varepsilon j}(t)\Big), \quad \boldsymbol{x}\in\omega_\varepsilon^j, \tag{4.1}$$

where B_ε is an integral operator [see (3.5)].

Equation (4.1) implies the following relation [similarly to (3.6)]

$$F_3^{\varepsilon j}(t) = \frac{\pi E}{1-\nu^2}\,c_j^\varepsilon\Big[\delta_0^j(t) + \frac{\Lambda_0}{c_2}\,\frac{1-\nu^2}{\pi E}\,\dot F_3^{\varepsilon j}(t) - \sum_{k\neq j}\Big(\frac{1-\nu^2}{\pi E}\,\frac{F_3^{\varepsilon k}(t)}{d_{jk}} + g_3(t,d_{jk}) * \dot F_3^{\varepsilon j}(t)\Big)\Big] \qquad (j=1,2,\dots,N). \tag{4.2}$$

Here c_j^ε is the sliding capacity of the punch ω_ε^j. For a system of elliptic punches, the quantity c_j^ε is given by formula (3.7). In the particular case of a system of circular punches, relation (4.2) agrees with relation (6) in [2] obtained by a different method. Equation (4.2) is an extension of the solution of the quasistatic contact problem to a system of a large number of arbitrary small punches obtained in [14].

Equation (4.2) can be written as

$$F_3^{\varepsilon j}(t) = \frac{\pi E}{1-\nu^2}\,c_j^\varepsilon\Big(\delta_0^j(t) + \frac{\Lambda_0}{c_2}\,\frac{1-\nu^2}{\pi E}\,\dot F_3^{\varepsilon j}(t) - \sum_{k\neq j}\tilde g_3(t,d_{jk}) * \dot F_3^{\varepsilon k}(t)\Big), \tag{4.3}$$

where

$$\frac{\pi E}{1-\nu^2}\,d_{jk}\tilde g_3(t,d_{jk}) * \dot F_3^{\varepsilon k}(t) = \Big(\frac{1}{2} + f_{12}(\beta^{-1})\Big)F_3^{\varepsilon k}\Big(t-\frac{d_{jk}}{c_1}\Big) + \Big(\frac{1}{2} - f_{12}(1)\Big)F_3^{\varepsilon k}\Big(t-\frac{d_{jk}}{c_2}\Big) + \int\limits_{\beta^{-1}}^{1} f_{12}'(\tau)F_3^{\varepsilon k}\Big(t-\frac{d_{jk}}{c_2}\tau\Big)d\tau - \int\limits_{1}^{\gamma} f_{23}(\tau)\,\frac{dF_3^{\varepsilon k}}{d\tau}\Big(t-\frac{d_{jk}}{c_2}\tau\Big)d\tau.$$

Let us consider Eqs. (4.3) for $j = 1,2,\dots,N$ as a system of equations for the unknown forces $F_3^{\varepsilon 1}(t),\dots,F_3^{\varepsilon N}(t)$. With the same accuracy as Eq. (4.3) was derived, we can set

$$\dot F_3^{\varepsilon k}(t) = \frac{\pi E}{1-\nu^2}\,c_k^\varepsilon\dot\delta_0^k(t). \tag{4.4}$$

Substituting expression (4.4) into Eq. (4.3), we bring it to the form

$$-\frac{\Lambda_0}{c_2}\,\dot F_3^{\varepsilon j}(t) + \frac{1}{c_j^\varepsilon}\,F_3^{\varepsilon j}(t) = \frac{\pi E}{1-\nu^2}\Big\{\delta_0^j(t) - \sum_{k\neq j}\frac{\pi E}{1-\nu^2}\,c_k^\varepsilon\tilde g_3(t,d_{jk}) * \dot\delta_0^k(t)\Big\}. \tag{4.5}$$

Denoting by $\Delta_0^j(t)$ the expression in the braces on the right side of the differential equation (4.5), we write the particular solution of Eq. (4.5) obtained using the small parameter method in the form

$$\frac{1-\nu^2}{\pi E}\,F_3^{\varepsilon j}(t) = c_j^\varepsilon\sum_{n=0}^{\infty}\Big(\frac{\Lambda_0 c_j^\varepsilon}{c_2}\Big)^n\,\frac{d^n\Delta_0^j}{dt^n}(t). \tag{4.6}$$

In the infinite sum (4.6), we retain only terms of order ε compared to unity. Thus, we finally obtain

$$\frac{1-\nu^2}{\pi E}\,F_3^{\varepsilon j}(t) = c_j^\varepsilon\Delta_0^j(t) + \frac{(c_j^\varepsilon)^2\Lambda_0}{c_2}\,\dot\Delta_0^j(t)$$

or [dropping terms $O(\varepsilon^2)$],

$$\frac{1-\nu^2}{\pi E}\,\frac{1}{c_j^\varepsilon}\,F_3^{\varepsilon j}(t) = \delta_0^j(t) + \frac{c_j^\varepsilon\Lambda_0}{c_2}\,\dot\delta_0^j(t) - \sum_{k\neq j}\frac{\pi E}{1-\nu^2}\,c_k^\varepsilon\tilde g_3(t,d_{jk}) * \dot\delta_0^k(t).$$

5. Example. Using the first-order asymptotic model constructed above, we consider the contact dynamic problem of steady-state vibrations of a system of two circular punches on an elastic half-space. Let one of the punches be acted upon by a periodic vertical force $P\exp(i\omega t)$. The vertical displacements of the punches $\delta_0^1(t)$ and $\delta_0^2(t)$ are determined from the system of equations

$$\begin{aligned} m_1\ddot{\delta}_0^1(t) &= -F_1(t) + P\exp(i\omega t), \\ m_2\ddot{\delta}_0^2(t) &= -F_2(t). \end{aligned} \tag{5.1}$$

According to formula (4.3), the reactions of the elastic medium $F_1(t)$ and $F_2(t)$ are calculated as follows:

$$\begin{aligned} F_1(t) &= \frac{\pi E}{1-\nu^2}\, c_1\Big(\delta_0^1(t) + \frac{\Lambda_0}{c_2}\,\frac{1-\nu^2}{\pi E}\,\dot{F}_1(t) - \tilde{g}(t,d) * \dot{F}_2(t)\Big), \\ F_2(2) &= \frac{\pi E}{1-\nu^2}\, c_2\Big(\delta_0^2(t) + \frac{\Lambda_0}{c_2}\,\frac{1-\nu^2}{\pi E}\,\dot{F}_2(t) - \tilde{g}(t,d) * \dot{F}_1(t)\Big). \end{aligned} \tag{5.2}$$

Here

$$\frac{\pi E}{1-\nu^2}\, d\tilde{g}(t,d) * \dot{F}(t) = \Big(\frac{1}{2} + f_{12}(\beta^{-1})\Big)F(t - dc_1^{-1}) + \Big(\frac{1}{2} - f_{12}(1)\Big)F(t - dc_2^{-1})$$

$$+ \int\limits_{\beta^{-1}}^{1} f_{12}'(\tau)F(t - dc_2^{-1}\tau)\,d\tau + \int\limits_{1}^{\gamma} f_{23}(\tau)\,\frac{dF}{d\tau}(t - dc_2^{-1}\tau)\,d\tau.$$

For simplicity, the sliding capacities of the circular punches are considered equal to $c_1 = c_2 = 2a/\pi$.

We set

$$\delta_0^j(t) = A_j\exp(i\omega t), \qquad F_j(t) = B_j\exp(i\omega t) \qquad (j = 1, 2).$$

Having derived the equations for the complex amplitudes B_1 and B_2 of the reactions $F_1(t)$ and $F_2(t)$ from system (5.1), (5.2), we determine the complex amplitudes of the vertical displacements of the punches A_1 and A_2 at the specified frequency of the perturbing force ω.

Introducing the dimensionless frequency and amplitudes

$$\varkappa = \frac{a\omega}{c_2}, \qquad \tilde{A}_j = \frac{A_j}{a}, \qquad \tilde{B}_j = \frac{1-\nu^2}{\pi E}\,\frac{B_j}{a^2} \qquad (j = 1, 2),$$

we express the dimensionless complex amplitudes of the vertical displacements $\tilde{A}_1$ and $\tilde{A}_2$ in terms of the quantities $\tilde{B}_1$ and $\tilde{B}_2$:

$$\tilde{A}_1 = \frac{1}{\varkappa^2\mu_1}\,\tilde{B}_1 - \frac{1}{\varkappa^2\mu_1}, \qquad \tilde{A}_2 = \frac{1}{\varkappa^2\mu_2}\,\tilde{B}_2. \tag{5.3}$$

In this case, the equations for the complex amplitudes of the reactions $\tilde{B}_1$ and $\tilde{B}_2$ become

$$\begin{aligned} \Big(\frac{\pi}{2} - \frac{1}{\varkappa^2\mu_1} - i\varkappa\Lambda_0\Big)\tilde{B}_1 + \varepsilon\tilde{G}(\varkappa,\varepsilon)\tilde{B}_2 &= -\frac{1}{\varkappa^2\mu_1}\,\tilde{P}, \\ \varepsilon\tilde{G}(\varkappa,\varepsilon)\tilde{B}_1 + \Big(\frac{\pi}{2} - \frac{1}{\varkappa^2\mu_2} - i\varkappa\Lambda_0\Big)\tilde{B}_2 &= 0, \end{aligned} \tag{5.4}$$

where

$$\varepsilon = \frac{a}{d}, \qquad \mu_j = \frac{1-\nu^2}{\pi E}\,\frac{m_j c_2^2}{a^3}, \qquad \tilde{P} = \frac{1-\nu^2}{\pi E}\,\frac{1}{a^2}\,P,$$

$$\tilde{G}(\varkappa,\varepsilon) = \Big(\frac{1}{2} + f_{12}(\beta^{-1})\Big)\exp(-i\varepsilon^{-1}\beta^{-1}\varkappa) + \Big(\frac{1}{2} - f_{12}(1)\Big)\exp(-i\varepsilon^{-1}\varkappa)$$

$$+ \int\limits_{\beta^{-1}}^{1} f_{12}'(\tau)\exp(-i\varepsilon^{-1}\varkappa\tau)\,d\tau - i\varepsilon^{-1}\varkappa\int\limits_{1}^{\gamma} f_{23}(\tau)\exp(-i\varepsilon^{-1}\varkappa\tau)\,d\tau.$$

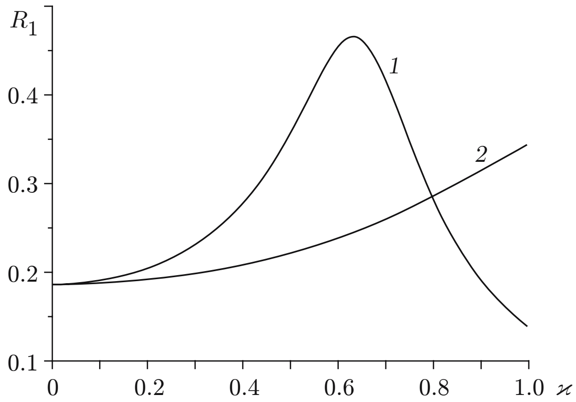

Fig. 1

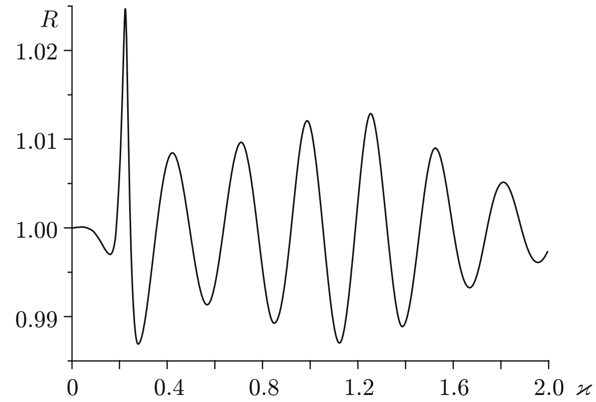

Fig. 2

Fig. 1. Vertical displacement amplitude of the first punch R_1 versus frequency $\varkappa$ in the absence of the second punch for $b_1 = 10$ (1) and 2 (2).

Fig. 2. Ratio R of the vertical displacement amplitudes of the first punch versus frequency $\varkappa$ ($b_1 = 2$ and $b_2 = 100$) in the presence and absence of the second punch.

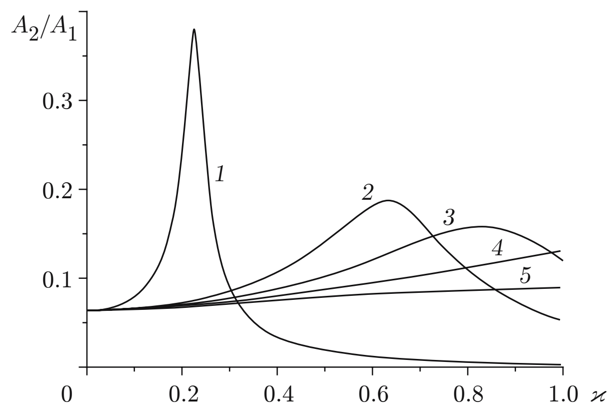

Fig. 3. Ratio of the vertical displacement amplitudes of the punches A_2/A_1 versus frequency $\varkappa$ for $b_2 = 100$ (1), 10 (2), 5 (3), 2 (4), and 0 (5).

Results of numerical calculations by formulas (5.3) and (5.4) are presented in Figs. 1–3. In this case, $\varepsilon = 0.1$, and the parameter b_j is given by the formula

$$b_j = \frac{2\pi}{1-\nu^2}\,\mu_j \qquad (j = 1, 2).$$

Figure 1 shows curves of the vertical displacement amplitude of the first punch $R_1 = A_1\mu a/P$ versus frequency $\varkappa$ in the absence of the second punch. Figure 2 shows the ratio R of the vertical displacement amplitude of the first punch in the presence and absence of the second punch for $b_1 = 2$ and $b_2 = 100$. From Fig. 2, it follows that the presence of the second punch has little effect on the vertical displacement amplitude of the first punch. Figure 3 shows a curve of the ratio of the vertical displacement amplitude of the punches versus frequency for various values of the parameter b_2. The calculation results presented in Figs. 1–3 are in good agreement with the results obtained in [15] using a different approximate method.

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